Asymptotic Efficiency of Quantum Hypothesis Testing: The Quantum Chernoff Bound

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Classical Chernoff Bound

Asymptotic error rates in hypothesis testing: Chernoff (1952), Sanov (1957) and Hoeffding (1965)

- Question is to choose between two possible explanations (or models) called Hypothesis $\Longrightarrow H_0, H_1$

- Decision is based on a set of data collected from observations.

Example:

Deciding whether a patient is healthy (hypothesis H_0) or has certain disease (hypothesis H_1) based on some clinical tests.

H_0 ⟹ working hypothesis or null hypothesis;

H_1 ⟹ the alternative hypothesis.

Two types of errors:

(1) the rejection of a true null hypothesis (wrongly concluding that a healthy patient has the disease)

probability ⟹ $p(1 \mid H_0) = p_0(1)$

(2) the acceptance of a false null hypothesis (failure to diagnose the disease)

probability ⟹ $p(0 \mid H_1) = p_1(0)$

Minimizing the errors

- Common approach: Minimize one of the errors by keeping the other bounded by a constant (depending on the number of observations)
- Another approach (Baysean-like) : Minimize a linear combination of two error probabilities

$$P_e = \min[\pi_0 \; p(1 \mid H_0) + \pi_1 \; p(0 \mid H_1)]$$
$$= \min[\pi_0 \; p_0(1) + \pi_1 \; p_1(0)]$$

$\pi_0, \quad \pi_1$ ⟹ *a priori* probabilities assigned to the occurrence of each hypothesis.

With N optimal tests, the probability of error P_e declines exponentially as (considering equal *a priori* probabilities)

$$P_e \approx \mathrm{Exp}[-N\, C(p_0,p_1)],$$

$$C(p_0,p_1) = -\min_{s=[0,1]} \log \sum_{b=0,1} p_0^s(b)\, p_1^{1-s}(b) \qquad (1)$$

The so called “Chernoff information” or Chernoff distance $C(p_0,p_1)$ is expressed in terms of the Kullback-Leibler divergence

$$C(p_0,p_1) = K(p_{s^*} \| p_0) = K(p_{s^*} \| p_1)$$

$$p_{s^*}(b) = \frac{p_0^s(b)\, p_1^{1-s}(b)}{\sum_b p_0^s(b)\, p_1^{1-s}(b)},$$

$$K(p_{s^*} \| p_0) = \sum_b p_{s^*}(b) \log[\, p_0(b) / p_{s^*}(b)]$$

s^* is the value of $s=[0,1]$ that minimizes the right-hand side of (1).

Quantum Scenario

- ➢ States of quantum-mechanical objects are described by density matrices.
- ➢ A density matrix is a self-adjoint, nonnegative operator of a complex Hilbert space with a trace of 1.
- ➢ States are not directly observable: they can be measured -- the outcome of a measurement treated as a random variable.

- In the case of a countable number of outcomes, every measurement can be represented by a set $\{E_i\}, i = 1,2,..., k$ of nonnegative operators which are required to add up to the identity operator: $\sum_{i=1}^{k} E_i = I$

- Each operator E_i in the set corresponds to a particular outcome of the measurement.

$$\text{State} \rightarrow \rho$$

$$\text{Probability of outcome } i \rightarrow \mathrm{Tr}[\rho \mathrm{E_i}]$$

Projective measurements form an important subclass: E_i orthogonal projectors $E_i E_j = \delta_{ij} E_j$

$\delta_{ij} \rightarrow$ Kronecker delta symbol.

Suppose we are given a sample of N identical quantum states, which are either ρ_0 or ρ_1 with the prior probability 1/2. Task is to minimize the average probability of making an incorrect decision about the state by devising a system of measurements and a decision rule.

Take a two element POVM set: $\{E_0, E_1; E_0 + E_1 = I\}$

Single copy minimum error probability is given by

$$P_{e,Q}^{(1)} = \frac{1}{2}\min_{\{\mathrm{E}_0,\mathrm{E}_1\}}\left(\mathrm{Tr}[\rho_0 E_1] + \mathrm{Tr}[\rho_1 E_0]\right)$$

$$= \frac{1}{2}\min_{\{\mathrm{E}_0,\mathrm{E}_1\}}\left(1 - \mathrm{Tr}[(\rho_0 - \rho_1)E_0]\right)$$

$$= \frac{1}{2}\min_{\{\mathrm{E}_0,\mathrm{E}_1\}}\left(1 + \mathrm{Tr}[(\rho_0 - \rho_1)E_1]\right)$$

$$= \frac{1}{2}\min_{\{\mathrm{E}_0,\mathrm{E}_1\}}\left(1 - \frac{1}{2}\mathrm{Tr}[(\rho_0 - \rho_1)(E_0 - E_1)]\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\max_{\{\mathrm{E}_0,\mathrm{E}_1\}}\mathrm{Tr}[(\rho_0 - \rho_1)(E_0 - E_1)]\right)$$

Quantum Statistics ⟹ ability to vary distributions over outcomes by choosing appropriate measurements on given quantum states.

Choose

$$E_0 = \sum_{\alpha} \left| \psi_{+\alpha} \right\rangle \left\langle \psi_{+\alpha} \right|,$$

$$\downarrow$$

$$\text{Eigenvectors of } (\rho_0 - \rho_1) \text{ corresponding to}$$
$$\text{positive/negative eigenvalues}$$

$$\uparrow$$

$$E_1 = \sum_{\beta} \left| \psi_{-\beta} \right\rangle \left\langle \psi_{-\beta} \right|$$

………… so that

the maximum of

$$\max_{\{E_0,E_1\}} \mathrm{Tr}[(\rho_0-\rho_1)(E_0-E_1)] = \| \rho_0-\rho_1 \|$$

is obtained.

Tracenorm

Therefore, the minimum error probability in distinguishing the two states $\rho_0, \quad \rho_1$ takes the form

$$P_{e,Q}^{(1)} = \frac{1}{2}\left[1-\frac{1}{2}\| \rho_0-\rho_1 \|\right]$$

(Holevo-Helstrom result)

N copy error probability:

$$P_{e,Q}^{(N)} = \frac{1}{2}\left[1 - \frac{1}{2} \| \rho_0{}^{\otimes N} - \rho_1{}^{\otimes N} \|\right]$$

How does the error decline as N grows??

Finding the eigenvalues of $\rho_0{}^{\otimes N} - \rho_1{}^{\otimes N}$
is a hard computational task --- as the dimensionality of the states grows rapidly with increasing sample size N

Some special cases

(1) Both the states to be discriminated are pure:

$$\rho_0 = |\psi_0\rangle\langle\psi_0|, \quad \rho_1 = |\psi_1\rangle\langle\psi_1|$$

N copy error probability is given by

$$P_{e,Q,\text{pure}}^{(N)} = \frac{1}{2}\left[1 - \frac{1}{2}\sqrt{1 - \left|\langle\psi_0|\psi_1\rangle\right|^{2N}}\right]$$

Asymptotical decline:

$$\lim_{N\to\infty} \frac{1}{N} \log P_{e,Q,\text{pure}}^{(N)} \approx 2 \log \left|\langle\psi_0 | \psi_1\rangle\right|$$

(2) If the states ρ_0 and ρ_1 commute, then classical error decline rate holds.

Bounds on error:

Any two positive operators A, B satisfy the inequality

$$[A^s B^{1-s}] \geq \frac{1}{2}\big[\mathrm{Tr}[A+B] - \| A - B \|_1 \big], \qquad 0 \leq s \leq 1$$

(Audenaert et. al., Phys. Rev. Lett. 98, 160501 (2007))

Choosing

$$A = \frac{1}{2}\rho_0^{\otimes N}, B = \frac{1}{2}\rho_1^{\otimes N}, \quad \text{we get}$$

$$\frac{1}{2}\mathrm{Tr}\left[\left(\rho_0^{\otimes N}\right)^s \left(\rho_1^{\otimes N}\right)^{1-s}\right] \geq \frac{1}{2}\left[1 - \frac{1}{2}\| \rho_0^{\otimes N} - \rho_1^{\otimes N} \|_1\right]$$

$$\text{or} \quad P^{(N)}{}_{e,Q} \leq P^{(N)}{}_{e,QCB} \equiv \min_{0 \leq s \leq 1}\left(\frac{1}{2}\mathrm{Tr}\left[\rho_0{}^s \rho_1{}^{1-s}\right]^N\right)$$

For s=1/2, gives Bhattacharya bound on error

$$P^{(N)}{}_{e,QCB} \equiv \min_{0\leq s\leq 1}\left(\frac{1}{2}\,\mathrm{Tr}\left[\rho_0{}^{s}\rho_1{}^{1-s}\right]^{N}\right)$$

Quantum Chernoff Bound

Reduces to the results on error in special cases

When only one of the states is pure, i.e., $\rho_1 = |\psi_1\rangle\langle\psi_1|$

$$P^{(N)}{}_{e,QCB} \equiv \frac{1}{2}\langle\psi_1|\rho_0|\psi_1\rangle^N = \frac{1}{2}[F(\rho_0,\rho_1)]^N$$

Fidelity

$$F(\rho_0,\rho_1) = \left(\mathrm{Tr}[\sqrt{\sqrt{\rho_1}\rho_0\sqrt{\rho_1}}]\right)^2$$

Upper and lower bounds on N-copy error probability

Fuch-Graaf:

$$[1-\sqrt{F(\rho_0,\rho_1)}] \le \frac{1}{2} \| \rho_0 - \rho_1 \|_1 \le \sqrt{1-F(\rho_0,\rho_1)};$$

$$F({\rho_0}^{\otimes N},{\rho_1}^{\otimes N}) = F^N(\rho_0,\rho_1)$$

$$\Rightarrow \frac{1}{2}[1-\sqrt{1-F^N(\rho_0,\rho_1)}] \le P_{e,Q}^{(N)} \le \sqrt{F^N(\rho_0,\rho_1)}$$

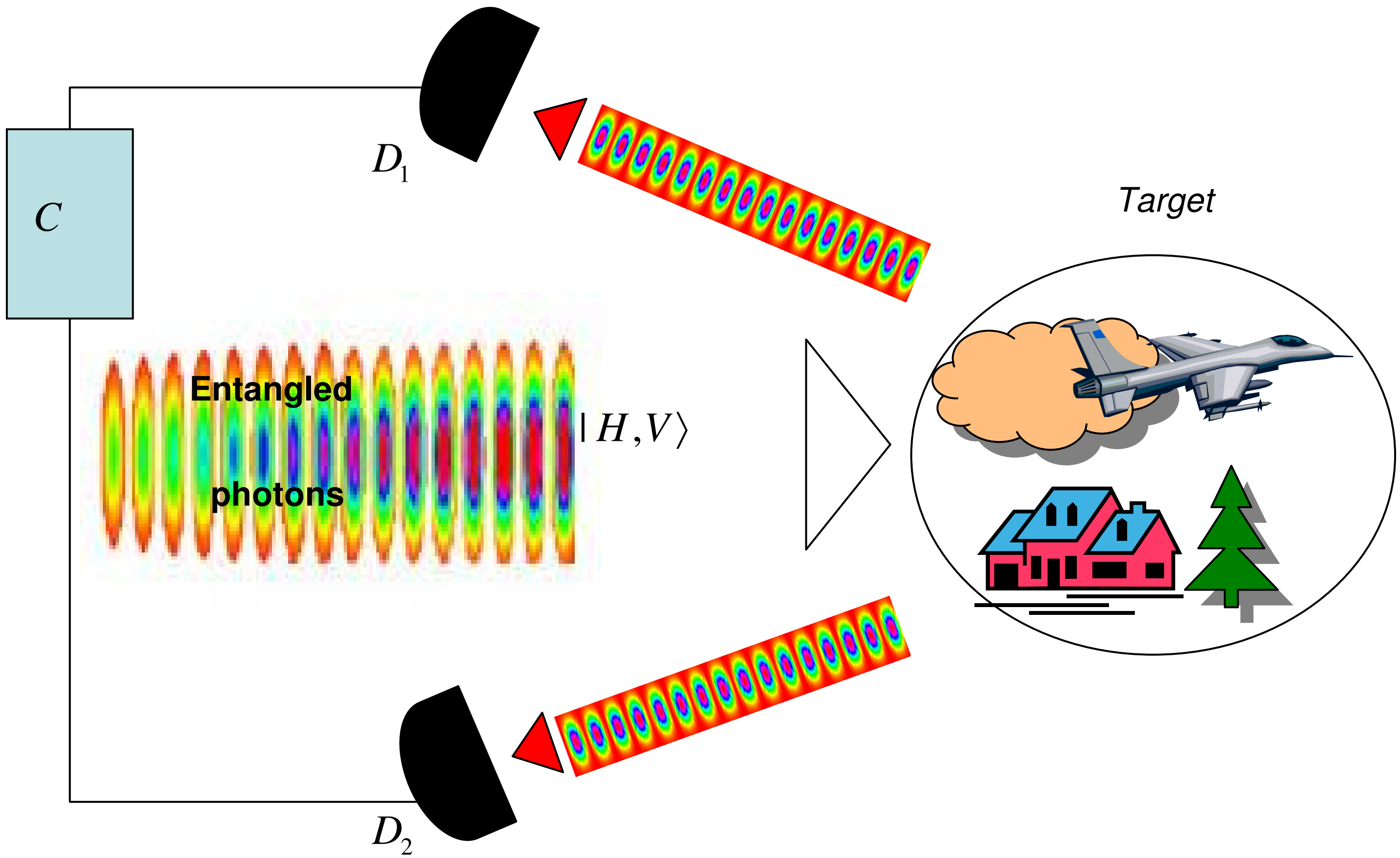
Quantum Bhattacharya Bounds

$$\frac{1}{2}\left(1-\sqrt{1-\left[\mathrm{Tr}[\rho_0{}^{1/2}\rho_1{}^{1/2}]\right]^{2N}}\right)\leq P_{e,Q}^{(N)}\leq\frac{1}{2}\left[\mathrm{Tr}[\rho_0{}^{1/2}\rho_1{}^{1/2}]\right]^{N}$$

$$\frac{1}{2}\left[\mathrm{Tr}[\rho_0{}^{1/2}\rho_1{}^{1/2}]\right]^{N}\geq P_{e,QCB}^{(N)}$$

Weaker upper bound

Quantum Target Detection



Let ρ_{in} be the input radiation state used to illuminate the target.

Hypothesis H_0 ⟹ **Target present:**

The state of the radiation, received at the detector: ρ_0

Hypothesis H_1 ⟹ **Target absent:**

The state of the radiation, received at the detector: ρ_1

Quantum target Detection $\Longrightarrow$ Ability to distinguish between the states ρ_0, ρ_1
(i.e., choose between the Hypotheses H_0, H_1)

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Enhanced Sensitivity of Photodetection via Quantum Illumination

Seth Lloyd

The use of quantum-mechanically entangled light to illuminate objects can provide substantial enhancements over unentangled light for detecting and imaging those objects in the presence of high levels of noise and loss.

Quantum Illumination with Gaussian States

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An optical transmitter irradiates a target region containing a bright thermal-noise bath in which a low-reflectivity object might be embedded. The light received from this region is used to decide whether the object is present or absent. The performance achieved using a coherent-state transmitter is compared with that of a quantum-illumination transmitter, i.e., one that employs the signal beam obtained from spontaneous parametric down-conversion. By making the optimum joint measurement on the light received from the target region together with the retained spontaneous parametric down-conversion idler beam, the quantum-illumination system realizes a 6 dB advantage in the error-probability exponent over the optimum reception coherent-state system. This advantage accrues despite there being no entanglement between the light collected from the target region and the retained idler beam.

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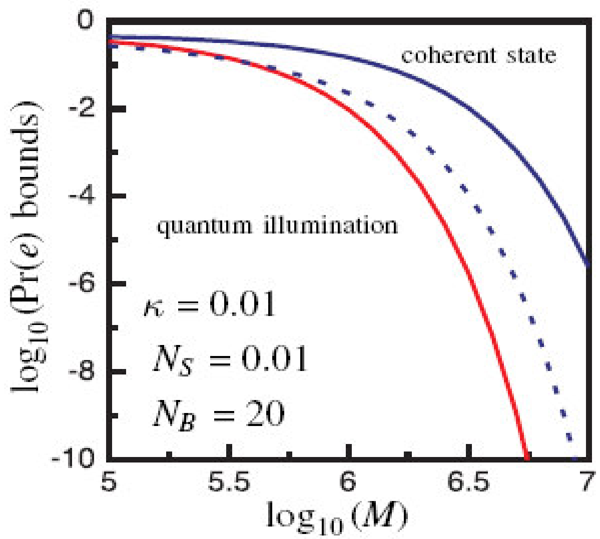

FIG. 1 (color online). Upper bounds (solid curves) on the target-detection error probabilities for coherent-state (Chernoff bound) and quantum-illumination (Bhattacharyya bound) transmitters with M transmitted modes each with $N_S = 0.01$ photons on average when $\kappa = 0.01$ and $N_B = 20$. Also shown is the lower bound (dashed curve) for the coherent-state case, which (see below) also applies to *all* classical-state transmitters with $\sum_{m=1}^{M} \langle \hat{a}_{S_m}^{\dagger} \hat{a}_{S_m} \rangle = MN_S$. For large M, the classical-state lower bound exceeds the quantum-illumination upper bound.

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Quantum target detection using entangled photons

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We investigate performances of pure continuous variable states in discriminating thermal and identity channels by comparing their M-copy error-probability bounds. This offers us a simplified mathematical analysis for quantum target detection with slightly modified features: the object—if it is present—perfectly reflects the signal beam irradiating it, while thermal noise photons are returned to the receiver in its absence. This model facilitates us to obtain analytic results on error-probability bounds, i.e., the quantum Chernoff bound and the lower bound constructed from the Bhattacharya bound on M-copy discrimination error probabilities of some important quantum states, like photon number states, N-photon maximally entangled (N00N) states, coherent states and the entangled photons obtained from spontaneous parametric down conversion (SPDC). Comparing the M-copy error-bounds, we identify that path-entangled states indeed offer enhanced sensitivity than the photon number state system, when average signal photon number is small compared to the thermal noise level. However, in the high signal-to-noise scenario, N00N states fail to be advantageous than the photon number states. Entangled SPDC photon pairs too outperform conventional coherent state system in the low signal-to-noise case. On the other hand, conventional coherent state system surpasses the performance sensitivity offered by entangled photon pair, when the signal intensity is much above that of thermal noise. We find an analogous performance regime in the lossy target detection (where the target is modeled as a weakly reflecting object) in a high signal-to-noise scenario.

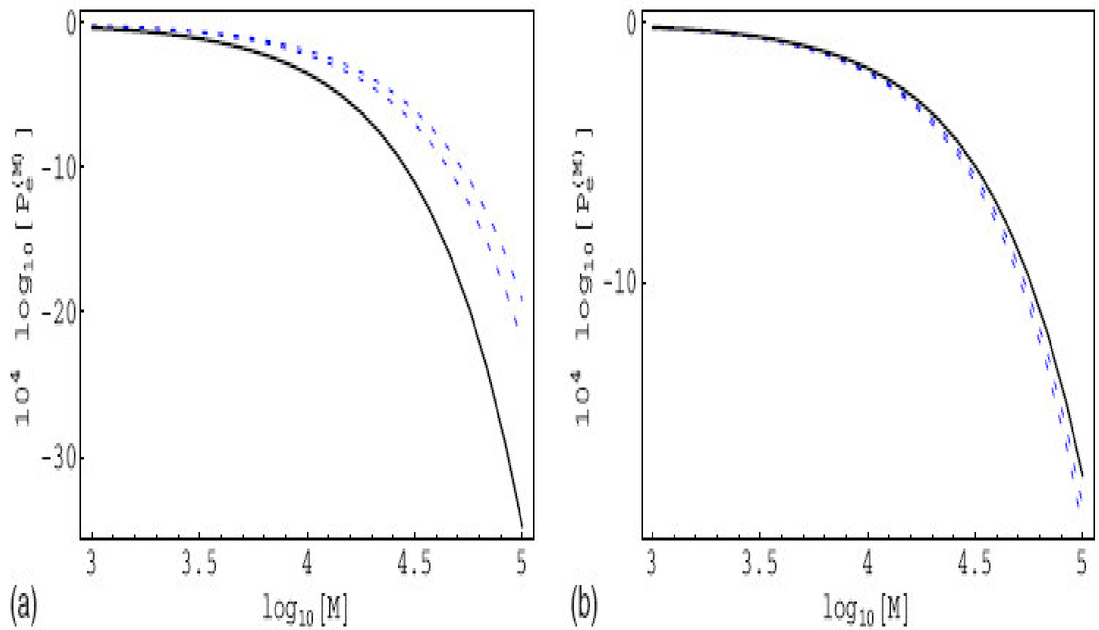

FIG. 1. (Color online) Upper, lower bounds (dashed curves) on M-copy error probability with N00N states and photon number state's error probability (solid curve) for a thermal noise $\beta=0.05$; photon numbers in (a) $n=100$ and in (b) $n=20$. The lower bound lies above the number state error probability in (a) implying that N00N states are *not* advantageous over photon number states. But, with smaller number of photons [as illustrated in (b)], entangled N00N states indeed offer an enhanced sensitivity over number state system.

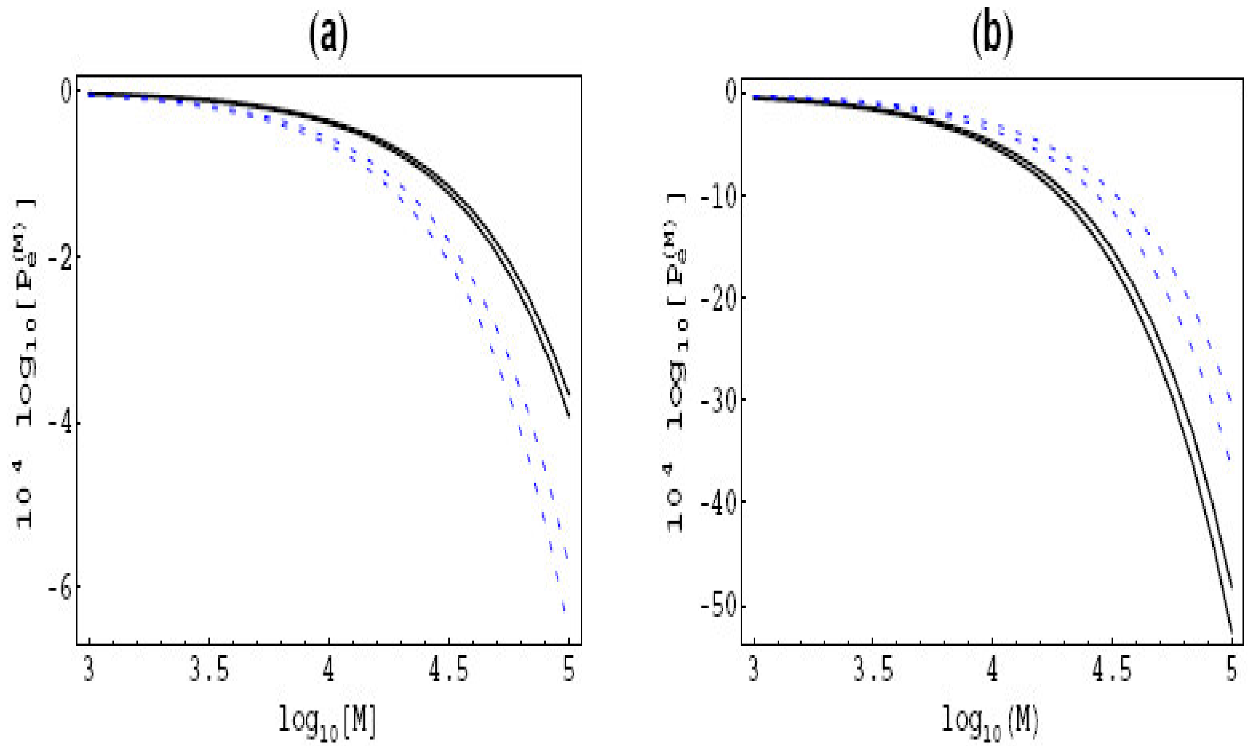

FIG. 2: (color online) Logarithms of upper and lower bounds (dashed curves) on M-shot error-probability with entangled photon pairs from SPDC source and that of coherent state system (solid curves) for (a) thermal noise $N_B = 0.75$ and $N_S = 0.5$ and in (b) $N_B = 2$, $N_S = 30$, plotted as a function of $\log_{10}[M]$. The target detection with $N_S < N_B$ in (a) is illustrative of the regime where entangled photon pairs show enhanced performance sensitivity over coherent light. But, it is seen from (b) that when $N_S >> N_B$ coherent state system is more advantageous than entangled SPDC photon pairs.

Entangled states do reveal enhanced performance sensitivity over unentangled ones in quantum target detection in certain regimes

-- identified with the help of Quantum Chernoff bound on M-copy error probabilities

Thank you